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SCATTERING AMPLITUDES AND THE CPT THEOREM IN STRING THEORY

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Abstract

We discuss the role of the CPT transformation in first quantized string theories, both on the world-sheet and in the space-time. We explicitly show that the space-time CPT theorem holds for all first-quantized (perturbative) string theories in a Minkowski background of even dimension $D > 2$.

1 Introduction

The Scattering Matrix is a central object in String Theory, even more than in Field Theory since at the moment we lack a full lagrangian formulation of String Theory and we mostly work in a first-quantized formalism. Indeed, roughly speaking a first-quantized perturbative String model is built by giving a 2d Conformal Field Theory (CFT), a GSO projection of the space-time spectrum (or a modular invariant one loop partition function) and the Polyakov formula for the scattering amplitudes.

Many interesting results have already been obtained by the explicit computation of string scattering amplitudes. Very well known are the results obtained

in Field Theory by comparing the string scattering amplitudes in the so-called “field-theory limit” and the same scattering amplitudes directly computed in Field Theory. Even so, a lot of work is still to be done. Indeed, up to now mainly tree level (genus zero) scattering amplitudes and one-loop (genus one) space-time bosonic scattering amplitudes have been computed. In ref. [2] an example was given of a one-loop scattering amplitude involving external space-time fermions.

Beside the technical problems which arise in the computation of scattering amplitudes involving space-time fermions and in the computation of multi-loop scattering amplitudes, we also need to understand better the general properties of the string scattering amplitudes. For example we know that the string scattering amplitudes have different analytical properties than the field theory scattering amplitudes. On the other hand, we do not know exactly how unitarity, causality,¹ CPT and other similar common and well-known properties in Field Theory, manifest themselves at the level of theta functions, prime forms and the other mathematical objects which explicitly form the integrand of a string scattering amplitude. Of course, the knowledge of this can also help us in doing the explicit computations.

Here we will report on some results that we have obtained concerning the role of the CPT transformation, on the world-sheet and in the space-time, in first-quantized (perturbative) string theories. We will first briefly review the definition of the CPT transformation and the statement of the CPT theorem in Field Theory. We will then introduce an explicit formulation of the String Scattering amplitudes and discuss some of its properties, including the role of the CPT transformation on the 2d world-sheet. We will finally formulate a space-time CPT transformation and show that for first-quantized string models on a Minkowski background the CPT theorem holds true.

2 CPT in Field Theory

In Field Theory the general properties on which the CPT theorem is based are: [1]

- Lorentz invariance;
- The energy is positive definite and there exists a Poincaré-invariant vacuum, unique up to a phase factor;
- Local commutativity, i.e. field operators at space-like separations either commute or anti-commute.

These general properties imply the validity of the spin-statistics relation (i.e. fields of integer (half-integer) spin are quantized with respect to Bose (Fermi) statistics) in $D > 2$. Then the spin-statistics relation, together with Lorentz invariance, imply the CPT theorem. Indeed it is quite easy to check at the

¹The concept of causality in string theory is not quite well understood. Here we refer to the generally accepted definition for which a scattered wave should not reach the detector before the incident wave strikes the target.

level of Lagrangian field theory that the spin-statistics relation and Lorentz invariance imply the CPT theorem.

We formulate the CPT transformation as the combination of Pauli's Strong Reflection (SR) transformation and Hermitean Conjugation (HC): $C + P + T = SR + HC$. The CPT transformation is anti-linear and is implemented by an anti-unitary operator Θ .

Strong Reflection transforms any single field $\phi(\vec{x}, t)$ into $\phi(-\vec{x}, -t)$, times an appropriate phase factor. It also inverts the order of a product of operators (for this reason it is a symmetry of the operator algebra only). The basic scalar, fermionic and vector fields transform as follows:

$$\begin{aligned}\phi(x) &\rightarrow \phi(-x) \\ \psi(x) &\rightarrow \varphi_{\text{SR}} \gamma^{D+1} \psi(-x) \quad \text{with} \quad (\varphi_{\text{SR}})^2 = -(-1)^{D/2} \\ \bar{\psi}(x) &\rightarrow -\varphi_{\text{SR}}^* \bar{\psi}(-x) \gamma^{D+1} \\ \phi^\mu(x) &\rightarrow -\phi^\mu(-x) .\end{aligned}\tag{1}$$

Here the phase φ_{SR} is fixed (up to a sign) by requiring consistency of the SR transformation with Hermitean Conjugation for real fields.

The formulation of CPT as the combination of SR and HC turns out to be convenient because, as we shall see, it can be adapted even to string theory.

As well known, the $C + P + T$ transformation acts on a state as ²

$$|p\rangle = |p, \eta, \{\lambda\}\rangle \rightarrow |\rho^{\text{CPT}}\rangle = \varphi_{\text{CPT}}(\eta, \{\lambda\}) |p, -\eta, \{-\lambda\}\rangle ,\tag{2}$$

which, of course, guarantees that in field theory for each particle there exists the corresponding anti-particle (with opposite helicity).

For a scattering amplitude the statement of SR/CPT invariance, i.e. the CPT theorem, becomes

$$\begin{aligned}\langle \rho_1; \dots; \rho_{N_{\text{out}}}; in | S | \rho_{N_{\text{out}}+1}; \dots; \rho_N; in \rangle &= \\ &= \langle \rho_1^{\text{CPT}}; \dots; \rho_{N_{\text{out}}}^{\text{CPT}}; out | S^\dagger | \rho_{N_{\text{out}}+1}^{\text{CPT}}; \dots; \rho_N^{\text{CPT}}; out \rangle^* \\ &= \langle \rho_N^{\text{CPT}}; \dots; \rho_{N_{\text{out}}+1}^{\text{CPT}}; in | S | \rho_{N_{\text{out}}}^{\text{CPT}}; \dots; \rho_1^{\text{CPT}}; in \rangle .\end{aligned}\tag{3}$$

3 Scattering Amplitudes in String Theory

We formulate the string theory scattering amplitudes (the so-called Polyakov formula) in the operator formalism, in a form which mimics the field theory Lehmann-Symanzik-Zimmermann (LSZ) formula. [4] We consider string theories on a Minkowski background of even dimension $D > 2$ with metric $\text{diag}(-1, 1, \dots, 1)$. We define the T -matrix element by

$$\frac{\langle \rho_1, \dots, \rho_{N_{\text{out}}} | S | \rho_{N_{\text{out}}+1}, \dots, \rho_N \rangle_{\text{connected}}}{\prod_{i=1}^N (\langle \rho_i | \rho_i \rangle)^{1/2}} =\tag{4}$$

²where p is the momentum, η the helicity or its generalization in higher dimensions, and $\{\lambda\}$ a set of charges or other labels.

$$i(2\pi)^D \delta^D(p_1 + \dots p_{N_{\text{out}}} - p_{N_{\text{out}}+1} - \dots - p_N) \prod_{i=1}^N (2p_i^0 V)^{-1/2} \times \\ T(\rho_1; \dots; \rho_{N_{\text{out}}} | \rho_{N_{\text{out}}+1}; \dots; \rho_N) ,$$

where $|\rho\rangle = \lim_{\zeta, \bar{\zeta} \rightarrow 0} \mathcal{W}_{|\rho\rangle}(z = \zeta, \bar{z} = \bar{\zeta}) |0\rangle$ and $\mathcal{W}_{|\rho\rangle} = c\bar{c}\mathcal{V}_{|\rho\rangle}$ is a primary conformal field of dimension $\Delta = \bar{\Delta} = 0$. To write a completely explicit formula for the T -matrix element, we may consider a heterotic string model in the RNS formalism (but the formula can be quite easily generalized to any other first-quantized (perturbative) string model). Then

$$T(\rho_1; \dots; \rho_{N_{\text{out}}} | \rho_{N_{\text{out}}+1}; \dots; \rho_N) = \quad (5) \\ = \sum_{g=0}^{\infty} (-1)^{g-1} C_g \int \left(\prod_{I=1}^{3g-3+N} d^2 m^I \right) \prod_{\mu=1}^g \left(\sum_{\alpha_{\mu}, \beta_{\mu}} C_{\beta_{\mu}}^{\alpha_{\mu}} \right) \times \\ \left\langle \prod_{I=1}^{3g-3+N} (\eta_I | b) \prod_{i=1}^N c(z_i) \right\rangle^2 \left(\prod_{A=1}^{N_{\text{PCO}}} \Pi(w_A) \right) \times \\ \mathcal{V}_{|\rho_1\rangle}^{(q_1)}(z_1, \bar{z}_1) \dots \mathcal{V}_{|\rho_N\rangle}^{(q_N)}(z_N, \bar{z}_N) \rangle$$

where q_i is the superghost charge of the i^{th} vertex operator and Π is the picture changing operator, the total number of which is given by $N_{\text{PCO}} = 2g - 2 - \sum_{i=1}^N q_i$. (For more details on the notation, see ref. [4].)

We should mention that to formulate the scattering amplitude in a Minkowski background one has to build the 2d spin-fields describing the space-time fermions with the required signature. This has been done in ref. [3].

The next step in a careful formulation of this scattering amplitude is the normalization of the amplitude itself and of all vertex operators. In a first-quantized formalism, the normalization does not come automatically as for example from the Dyson formula in Field Theory, and one has to use some general physical principle to fix it. One may fix the overall normalization C_g at any genus by considering the scattering of two gravitons in the Regge regime (high center of mass energy and low transfer momentum) and require the eikonal resummation of the amplitude. Analogously, we fixed the normalization of all vertex operators describing incoming and outgoing particles by requiring that the universal part of the absorption/emission at genus zero of a very low energy graviton assumes the form required by the principle of equivalence. The results are explicitly given in ref. [4].

One key point in fixing the normalization of the vertex operators is the relation between the vertex operator $\mathcal{W}_{|\rho\rangle}$ describing an incoming particle and the vertex operator $\mathcal{W}_{\langle\rho|}$ describing the same particle but outgoing. In field theory this relation is quite simply given by Hermitean Conjugation, but in string theory it turns out to be given by the CPT transformation on the world-sheet: [6]

$$\mathcal{W}_{\langle\rho|}(z = \zeta, \bar{z} = \bar{\zeta}) = (-1)^{q+1} (\mathcal{W}_{|\rho\rangle}(z = \zeta^*, \bar{z} = \bar{\zeta}^*))^{\text{WS-CPT}} \quad (6)$$

where q is the superghosts charge (integer for space-time bosons and half-integer for space-time fermions).

The World-Sheet CPT transformation is defined on the sphere as the combination $\text{WS-CPT} = \text{BPZ} + \text{HC}$, where BPZ is the Belavin-Polyakov-Zamolodchikov (BPZ) transformation [7] $z \rightarrow 1/z$. Since z is related to cylindrical coordinates by $z = \exp(\tau + i\sigma)$, BPZ is seen to map (τ, σ) into $(-\tau, -\sigma)$. From this point of view, BPZ is very much like SR. However, unlike SR, BPZ does *not* invert the order of operators and for this reason WS-CPT differs from the ordinary CPT transformation defined as $\text{CPT} = \text{SR} + \text{HC}$. Even so, WS-CPT leads to a World-Sheet CPT Theorem, [6] which follows immediately from the fact that the BPZ transformation is a global conformal diffeomorphism on the sphere and which is therefore valid for any conformal field theory on the sphere, even for ghosts and superghosts (which do not satisfy spin-statistics) and for spin-fields (which are non-local).

The world-sheet CPT transformation and the world-sheet CPT theorem can be extended to higher genus surfaces by means of sewing, [9] the detailed procedure of which is described in ref. [6]. Here it is sufficient to recall that the WS-CPT theorem for 2d CFTs on a genus g Riemann surface, when applied to eq. 5, leads to the following *formal* hermiticity property of the string scattering amplitudes:

$$(T(\rho_N; \dots | \dots; \rho_1))^* = T(\rho_1; \dots | \dots; \rho_N) . \quad (7)$$

The proof of this equation is only *formal* because the integral over the moduli in eq. 5 is not always convergent, and the regularization of the divergencies gives rise to the imaginary part of the scattering amplitude required by unitarity. Several regularization procedures (of varying generality) have already appeared in the literature, [10] [11] even if an analytical prescription similar to the Feynman $i\epsilon$ prescription has not yet appeared.³

4 The Space-Time CPT Theorem in String Theory

Recently there has been some interest in the question of possible space-time CPT non-conservation in string theory. Indeed, some mechanisms have been proposed that would lead to CPT-breaking effects that might be detected in the next generation of experiments. [13] [12] Moreover, the interest in studying CPT in String Theory is also in understanding better the role of space-time CPT in presence of quantum gravity.

Not too much is known and published on the space-time CPT properties of string theory. Sonoda [8] discussed and proved the space-time CPT theorem at the level of string perturbation theory for ten-dimensional heterotic strings in a Minkowski background. Kostelecky and Potting [13] proved the *dynamical* CPT invariance of the open bosonic and super string field theories, formulated in flat backgrounds –but they also suggested a method whereby CPT might be

³But see ref. [11] for a first step in this direction.

broken *spontaneously*, based on the possibility of a CPT non-invariant ground state.

Here we will briefly show that the space-time CPT theorem holds true for any first-quantized string theory on a Minkowski background in even dimensions $D > 2$. [5]

As in lagrangian Field Theory, we assume Lorentz invariance and the spin-statistics relation. More explicitly, we consider first-quantized string models in a Minkowski background, where any two vertex operators describing physical states of half-integer space-time spin anti-commute, whereas any other pair of physical vertex operators commute. This ensures that the T -matrix elements given by eq. 5 have standard spin-statistics properties under the exchange of any pair of external states.

We define the space-time CPT transformation as a map on the world-sheet, i.e. on the conformal fields. In a generic first-quantized string theory on a Minkowski background in $D > 2$ dimensions, the space-time coordinates are described by D bosonic *free* world-sheet fields X_μ and, when there is world-sheet supersymmetry, by their world-sheet superpartners ψ_μ . We define the string Strong Reflection as the transformation which maps

$$X^\mu \rightarrow -X^\mu \quad \psi^\mu \rightarrow -\psi^\mu , \quad (8)$$

leaves all other world-sheet fields invariant and does *not* invert the order of the operators. For consistency the spin-fields describing space-time fermions must transform as

$$S_A \rightarrow \varphi_{\text{SR}} (\Gamma^{D+1})_A^B S_B . \quad (9)$$

The world-sheet action, BRST current and GSO projection conditions are invariant under this transformation.

In field theory, the CPT transformation is defined as the combination of SR and HC. In string theory we define the space-time CPT transformation as the combination of the string SR transformation defined above and the map (6) between an incoming and outgoing vertex operator, i.e. the world-sheet CPT transformation: Space-Time CPT = string SR + $(\mathcal{W}_{|\rho\rangle} \rightarrow \mathcal{W}_{\langle\rho|})$

$$\mathcal{W}_{|\rho\rangle} \rightarrow \mathcal{W}_{|\rho^{\text{CPT}}\rangle} \equiv (\mathcal{W}_{\langle\rho|})^{\text{SR}} = (-1)^{q+1} \left((\mathcal{W}_{|\rho\rangle})^{\text{WS-CPT}} \right)^{\text{SR}} . \quad (10)$$

Using the hypothesis of space-time Lorentz invariance, it is possible to prove the following identity on any genus g Riemann surface [5]

$$\langle (\mathcal{O}_1)^{\text{SR}}(z_1, \bar{z}_1) \dots (\mathcal{O}_N)^{\text{SR}}(z_N, \bar{z}_N) \rangle = (-1)^{N_{\text{FP}}} \langle \mathcal{O}_1(z_1, \bar{z}_1) \dots \mathcal{O}_N(z_N, \bar{z}_N) \rangle \quad (11)$$

where $2N_{\text{FP}}$ is the number of space-time spinorial fields. Then it holds

$$\begin{aligned} T(\rho_N^{\text{CPT}}; \dots; \rho_{N_{\text{out}}+1}^{\text{CPT}} | \rho_{N_{\text{out}}}^{\text{CPT}}; \dots; \rho_1^{\text{CPT}}) &= \sum \int \langle (\dots) \mathcal{W}_{\langle\rho_N^{\text{CPT}}|} \dots \mathcal{W}_{|\rho_1^{\text{CPT}}\rangle} \rangle \\ &= \sum \int \langle (\dots) (\mathcal{W}_{|\rho_N\rangle})^{\text{SR}} \dots (\mathcal{W}_{\langle\rho_1|})^{\text{SR}} \rangle \end{aligned}$$

$$\begin{aligned}
&= (-1)^{N_{\text{FP}}} \sum \int \langle (\dots) \mathcal{W}_{|\rho_N\rangle} \dots \mathcal{W}_{\langle \rho_1|} \rangle \\
&= T(\rho_1; \dots; \rho_{N_{\text{out}}} | \rho_{N_{\text{out}}+1}; \dots; \rho_N) ,
\end{aligned} \tag{12}$$

where in the last line we used the hypothesis of the standard space-time spin-statistics relation. Eq. 12 is the proof of the space-time CPT theorem (see eq. 3), once we show that the space-time CPT transformation we have defined acts in the correct way on the space-time labels. In other words, the question is: Given $|\rho\rangle = |k, \eta, \{\lambda\}\rangle$, do we have $|\rho^{\text{CPT}}\rangle \sim |k, -\eta, \{-\lambda\}\rangle$?

Consider for example a particle carrying a $U(1)$ charge λ . The charge λ is the eigenvalue of the zero mode of an hermitean Kač-Moody current J_Λ . Now, by definition J_Λ does not transform under string SR, but under world-sheet CPT $J_\Lambda \rightarrow -J_\Lambda$, so that under space-time CPT λ correctly changes sign. This argument can be generalized and extended also to the helicity, thus finally proving that the space-time CPT transformation we defined does indeed act in the correct way on the space-time labels. This concludes the proof of the space-time CPT theorem for first-quantized string theories in a Minkowski background of even dimension $D > 2$.

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